THEORY OF PARALLELS.

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It is well known that Euclid was obliged to make an assumption in establishing the doctrine of parallel lines. This assumption he made in the form of an axiom, and the attempts of his commentators to justify this assumption have been shown covertly to involve a still greater one, namely, in some form or other, the doctrine of direction of which that of parallels is a part. This has been made the excuse for adopting as a definition "lines which have the same direction" the use of which phrase, according to a strict construction, assumes that lines which make equal angles with (that is differ equally in direction from) one secant line make equal angles with any other secant line.

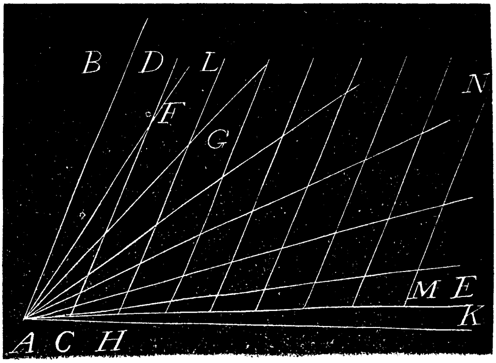
In 1834 a demonstration of Euclid's axiom appeared anonymously in Crelle's Journal, and it is my object in this note to reproduce this demonstration which seems to have been generally overlooked by writers of geometrical text-books, though apparently exactly what is needed to put the theory upon a perfectly sound basis. But before proceeding to the demonstration a few words on the exact position of Euclid's axiom in the theory.

It being impossible to establish directly that lines in a common plane which make equal angles with one secant line likewise make equal angles with any other secant (of which it would be an easy corrolary that such lines cannot meet), the idea of non-meeting lines is introduced and it is to be proved that this notion is co-extensive with that of making equal angles with a secant line. Consider now the four connected propositions whose logical form is

I.	Every case of	A is a case of	B.
II.	" " "	not A " " " "	not B.
III.	" " "	B " " " "	A.
IV.	" " "	not B " " " "	not A;

in which A is the notion of making equal angles with a secant line, and B is the notion of non-meeting how far so ever produced. I and IV follow logically one from the other; so likewise do II and III. In this case there is no difficulty in proving I and IV, the difficulty being in demonstrating II the "opposite" of I; namely that lines making unequal angles with a given line must meet. Now the "opposite" II (and hence the converse III) follows from I whenever the notion B is unique; that is if a case of A exists and has been proved to be a case of B, *there being but one case of B*; then *the* case of B is a case of A, for otherwise the case of A would

furnish a second case of B, which is impossible. For instance, when it has been proved that the bisector of an angle of a triangle cuts the base in the ratio of the adjacent sides, the "opposite" and converse follow at once, because there is but one point in which a line can be cut in a given ratio. Now Euclids assumption was the uniqueness of the notion B; that is, "through a given point but one line can be drawn parallel to (i. e. so as not to meet) a given line." The demonstration in Crelle of this proposition is a direct demonstration of II above, I being supposed previously established; it is substantially as follows:—



Let AB and CD be line making equal angles with the secant AE, so that AB and CL cannot meet, and let AF be any other line in the plane passing through A; then AF and CD will meet on that side of AE, on which the interior angles are less than two right angles.

From A draw a series of lines cutting off from FAE successive angles equal BAF, until we come to a line AK falling on the other side of AE, and let BAK contain n angles equal to BAF. Then, as the successive angles may be shown equal by coincidence, the space BAK (the lines being produced indefinitely) is n times the space BAF. Also draw a series of lines making the same angles with AE that AB makes, and cutting off from AE successive distances equal to AC, and let AM contain n distances equal to AC. Then the spaces $BACD$, $DCHL$, &c. may be shown equal by coincidence, therefore the space $BAMN$ equals n times the space $BACD$. Now the space BAK is greater than the space $BAMN$ therefore the space BAF is greater than $BACD$. Now if AF did not meet CD the space BAF would be less than the space $BACD$. Hence AF will if sufficiently produced meet CD.

PERFECT CUBES.

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THE object of this article is to resolve the general cubic polynomial $ax^3 + bx^2 + cx + d$, into special forms that may be rendered perfect cubes for particular values of x.